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Winding number locking on a two-dimensional torus: Synchronization of quasiperiodic motions

V. Anishchenko* and S. Nikolaev
Institute of Nonlinear Dynamics, Department of Physics, Saratov State University, 410012 Saratov, Russia

J. Kurths
Group of Nonlinear Dynamics, Institute of Physics, University of Potsdam, D-14415, Potsdam, Germany
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We propose a new autonomous dynamical system of dimension $N=4$ that demonstrates the regime of stable two-frequency motions and period-doubling bifurcations of a two-dimensional torus. It is shown that the period-doubling bifurcation of the two-dimensional torus is not followed by the resonance phenomenon, and the two-dimensional ergodic torus undergoes a period-doubling bifurcation. The interaction of two generators is also analyzed. The phenomenon of external and mutual synchronization of two-frequency oscillations is observed, for which winding number locking on a two-dimensional torus takes place.

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I. INTRODUCTION

Quasiperiodic motions and their bifurcations have, over the years, been a subject of research in nonlinear dynamics and turbulence. After publication of papers [1,2], the interest in phenomena of quasiperiodic motions with incommensurate frequencies ω_i, $i=1,2,3,\ldots,n$, $n\geqslant 4$ has decreased, as the instability of motions on three- and four-dimensional tori was proved. But there are some works pointing out the existence of stable motions with four and six independent frequencies [3]. So, the Landau-Hopf hypothesis about the transition to chaos through a high-dimensional torus must not be ruled out [4,5]. Furthermore, there is a large number of open problems connected to the properties of a two-dimensional torus. We mention here, e.g., the phenomenon of period-doubling bifurcation of a two-dimensional torus, which was reported for the first time in [6–8] and afterward by many other researchers. The problem on the bifurcational mechanism of period doubling of one of the periods of a two-dimensional torus is still unsolved [9–11]. There is a set of questions connected to synchronization phenomena of two-frequency oscillations, which is still opened. For example, is it possible to realize the phenomenon of winding number locking on a two-dimensional torus similar to the effect of frequency locking of a limit cycle? If it is, then what is the mechanism?

To answer the above-mentioned questions and others, it is reasonable to develop an autonomous basic model of a generator of two-frequency oscillations. Unlike periodically driven systems, the autonomous generator enables one to get answers to the questions mentioned above.

It is clear that a two-dimensional torus may be realized in a three-dimensional autonomous dynamical system, as it is shown, for example, in [12]. However, period-doubling bifurcations of a two-dimensional torus can be observed if the dimension of a system is at least $N=4$.

In the present paper we propose the simplest (from our point of view) autonomous four-dimensional dynamical system that demonstrates the regime of stable two-dimensional oscillations and period-doubling bifurcations of a two-dimensional torus. The period-doubling bifurcation of a two-dimensional torus is then studied in detail. It is shown that the ergodic (nonresonant) torus undergoes that bifurcation, and no resonances are observed in the bifurcation points. The phenomena of external and mutual synchronization of two-frequency motions are studied by analyzing the interaction of two unidirectionally and symmetrically coupled generators. The effect of winding number locking, which determines the complete synchronization of two-frequency motions is observed.

The generator developed will be used in future investigations of bifurcations of multi-frequency motions with a large ($n>4$) number of independent frequencies by investigating ensembles of such interacting generators.

II. GENERATOR OF QUASIPERIODIC MOTIONS WITH TWO INDEPENDENT FREQUENCIES

We consider the known model of the Anishchenko-Astakhov generator as a basic oscillator. Its scheme is shown in Fig. 1 [11]. The generator consists of the classic Van der Pol oscillator, into which the additional nonlinear inertial feedback is introduced. The generator represents a three-dimensional dynamical system with three parameters and is described by the following system of equations:

$$\dot{x} = mx + y - xz - dx^3,$$

$$\dot{y} = -x,$$

$$\dot{z} = -gz + g\Phi(x). \tag{1}$$

The first two equations of system (1) represent the Van der Pol oscillator for which $\dot{z}=0$ and $\Phi(x)=x^2$. As shown in [11], system (1) demonstrates the transition to chaos according to the Shilnikov theorem [13] provided that the nonlinear function $\Phi(x)$ is asymmetrical with respect to the variable x and has the following form:

*Electronic address: wadim@chaos.ssu.runnet.ru

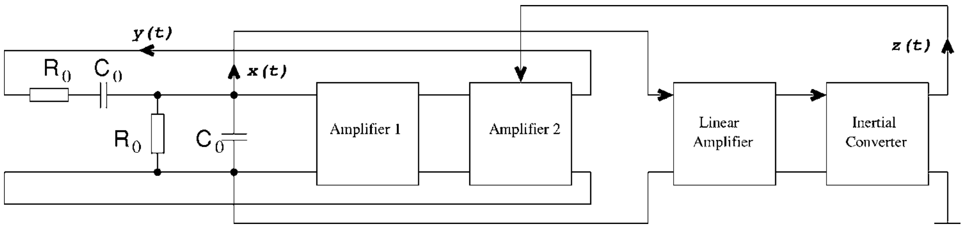

FIG. 1. Scheme of the Anishchenko-Astakhov generator.

$$\Phi(x) = I(x)x^2, I(x) = \begin{cases} 1, x > 0 \\ 0, x \leqslant 0 \end{cases} . \tag{2}$$

Under the conditions of asymmetry (2) a saddle-focus separatrix loop exists in system (1), which results in realizing the regime of spiral chaos. From a physical point of view, that phenomenon is a result of the self-consistent influence of feedback on the amplifier. The feedback signal is determined by the third equation in system (1). For small values of the signal $x(t)$ amplitude, this influence is insignificant and system (1) generates a limit cycle. As the parameter of excitation m grows, the intensity of $x(t)$ increases, the amplitude of the feedback signal $z(t)$ also increases, and its influence on the amplification coefficient of the main amplifier becomes more essential. The system demonstrates a cascade of period-doubling bifurcations of limit cycles and transition to chaos.

Undamped quasiperiodic motions with two independent frequencies can be realized in a generator of periodic motions if the excitation parameter of its amplifier is periodically modulated. In the case of system (1) this regime can be provided by introducing an oscillatory circuit into additional feedback of the generator. The feedback signal $z(t)$ must represent oscillations of an independent frequency, which modulate the amplification coefficient and give rise to quasiperiodic motions. This idea has been applied in practice as follows.

We propose the schemes presented in Fig. 2. Figure 2(a) shows the scheme of an inertial cascade of additional feedback in the Anishchenko-Astakhov generator. The cascade serves as a RC chain that has the form of a one-dimensional differential equation [the third equation in system (1)]. The scheme of the new inertial cascade shown in Fig. 2(b) includes an oscillatory circuit characterized by its own resonance frequency. The scheme in Fig. 2(b) is described by the following equations:

$$\dot{z} = \varphi,$$

$$\dot{\varphi} = -\gamma\varphi + \gamma\Phi(x) - gz, \tag{3}$$

where γ is the attenuation parameter, and g is the parameter defining the normalized resonance frequency of the new filter. One can see that Eqs. (3) describe a dissipative oscillatory circuit in the regime of forced oscillations

$$\ddot{z} + \gamma\dot{z} + gz = \gamma\Phi(x). \tag{4}$$

Next we introduce a new feedback into generator (1) so that the controlling signal of the feedback satisfies the condition $\dot{z}(t) = \varphi(t)$. Then the equations of the new generator take the form [14]

$$\dot{x} = mx + y - x\varphi - dx^3,$$

$$\dot{y} = -x,$$

$$\dot{z} = \varphi,$$

$$\dot{\varphi} = -\gamma\varphi + \gamma\Phi(x) - gz. \tag{5}$$

System (5) is a four-dimensional nonlinear dissipative dynamical system and is characterized by four controlling parameters: m is the excitation parameter, d is the parameter of nonlinear dissipation, γ is the attenuation parameter, and g is the sluggishness parameter of the filter. Two of those parameters are most important: The parameter of excitation m of the generator and the parameter of sluggishness g that characterizes the resonance frequency of the filter.

For $\Phi(x)$ satisfying Eq. (2) the system (5) has solutions in the form of stable two-frequency oscillations. This regime is exemplified in Fig. 3. The regime of two-frequency motions is stable, i.e., serves as an attractor with a sufficiently large basin of attraction. For example, our computations have shown that the phase point reaches the vicinity of limit set (Fig. 3) from the following range of initial values of $x(0)$: $-3.1 \leqslant x(0) \leqslant 0.9$. Hereinafter, the lower index of frequency "f_{ij}" means the following: The first number equals to "0" for the carrying frequency, and equals to "1" for the modulation one, the second number is the number of the generator.

In Fig. 4 the bifurcational diagram of system (5) is shown on the plane of two control parameters m and g and for the fixed values of parameters $\gamma=0.2$ and $d=0.001$. The lines l_u and l_c on the bifurcational diagram were calculated by analyzing the full spectrum of Lyapunov exponents (the SIMPACK program was used) and all the other lines were

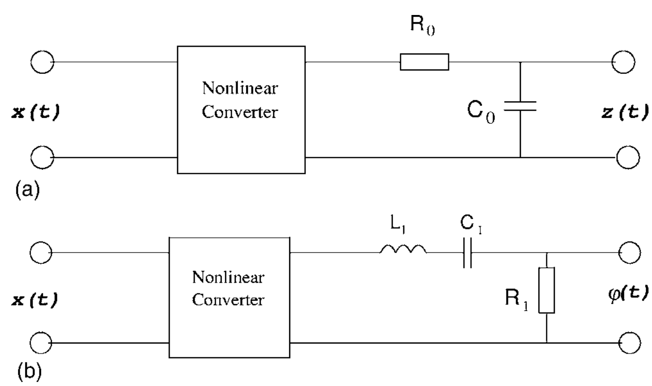

FIG. 2. Scheme of inertial cascade of additional feedback: (a) in the Anishchenko-Astakhov generator and (b) in the generator of quasiperiodic motions.

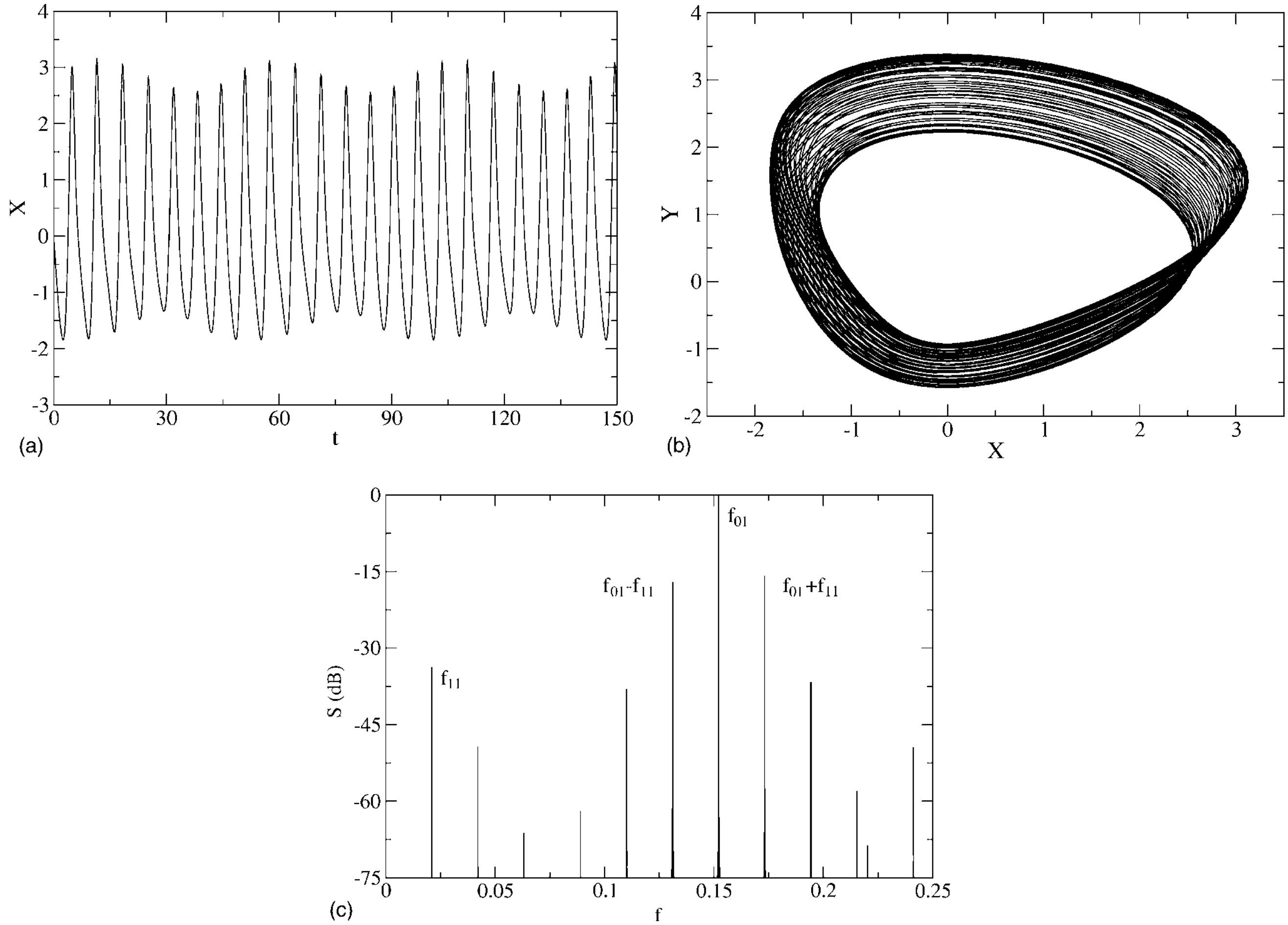

FIG. 3. Regime of quasiperiodic two-frequency motions: (a) time series; (b) projection of phase portrait; (c) power spectrum ($m=0.06$, $g=0.5$, $\gamma=0.2$, $d=0.001$). Initial conditions are $x(0)=-1.36$, $y(0)=1.31$, $z(0)=0.75$, $\varphi(0)=-0.77$.

constructed by using the program for bifurcational analysis AUTO 2000.

On the line $m=0$ a soft Andronov-Hopf bifurcation takes place, and as a result, a stable limit cycle T_0 is born. On the bifurcational line l_1 this cycle T_0 undergoes the period-doubling bifurcation. The new cycle emerged on line l_1 undergoes the same bifurcation on line l_2 (Fig. 4). On the bifurcational line l_t a pair of complex-conjugate multipliers of the limit cycle T_0 reach a unit circumference, which results in a soft bifurcation of two-dimensional torus appearance ($\mu_{1,2}=\exp\pm j\theta$, Neimark bifurcation). Obviously, when moving along line l_t, an angle θ possesses various rational quantities corresponding to the resonance phenomena on the torus. Figure 4 exemplifies the resonance $\theta=1:4$ region that is separated by lines of saddle-node bifurcations of the resonance cycle on the torus l_r. Inside that region period-doubling bifurcations of that resonance cycle can be observed when the parameters are varied. That region flows into the point A of codimension 2. Above the line of torus birth l_t, the line l_u is shown. When crossing it bottom up, a transition to chaos is observed through the stability loss of quasiperiodic motions. The line l_c corresponds to a crisis (destruction) of the chaotic attractor that emerges on the line l_u. On the line l_{dc} a pair of saddle cycles merge and then disappear.

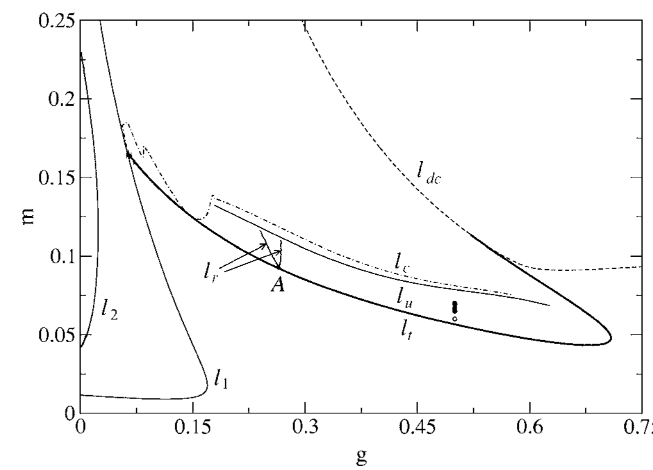

FIG. 4. Bifurcational diagram of regimes of generator ($\gamma=0.2$, $d=0.001$). $l_{1,2}$—lines of period-doubling bifurcations of cycles, l_t—bifurcational line of two-dimensional torus emergence, l_u—line of torus stability loss, l_c—line of chaotic attractor destruction, l_r—lines separating the resonance region on torus $1:4$, l_{dc}—line of multiple cycles, A—the codimension two point satisfying the condition $\theta=1:4$. "∘"—parameters values for the regimes in Fig. 3, "•"—parameters values for the regimes in Fig. 5.

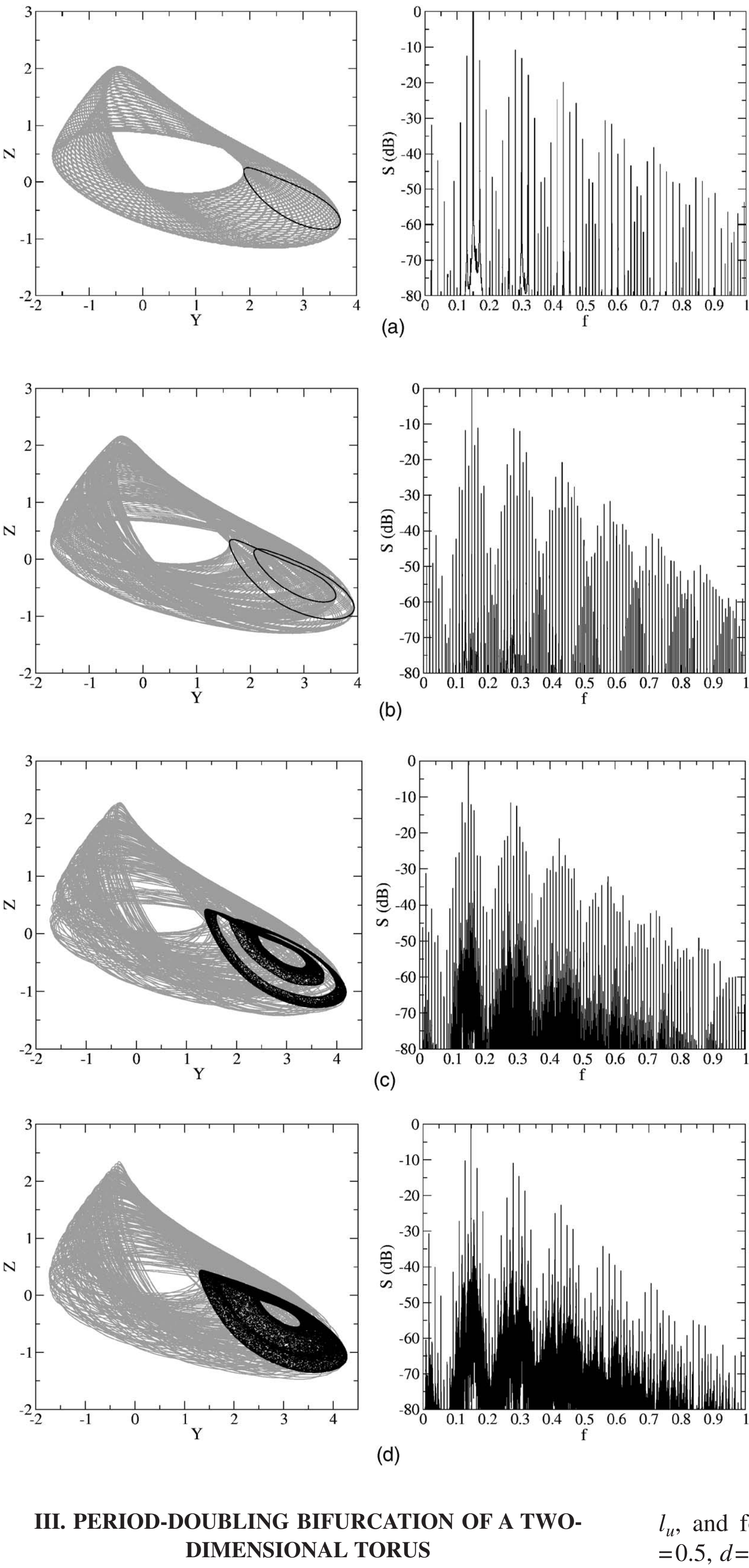

FIG. 5. Projections of attractors of system (5) and of their Poincare sections on the plane, and corresponding power spectra for different values of the parameter m and for the fixed values of all the other parameters d=0.001, γ=0.2, g=0.5: (a) m=0.065; (b) m=0.067; (c) m=0.0693; and (d) m=0.07.

III. PERIOD-DOUBLING BIFURCATION OF A TWO-DIMENSIONAL TORUS

Period-doubling bifurcations of a two-dimensional torus can be observed in the region between the lines l_t and l_u.

Now we analyze the evolution of the torus regime for various values of the parameter m in between the lines l_t and l_u, and for the fixed values of all the other parameters: g =0.5, d=0.001, and γ=0.2. In Figs. 5(a)–5(d) projections on the plane of phase variables (y,z) are shown for various attractors which appeared as a result of crossing the period-doubling bifurcation points. The observed changes in the structure of the Poincaré sections, in time series and in power

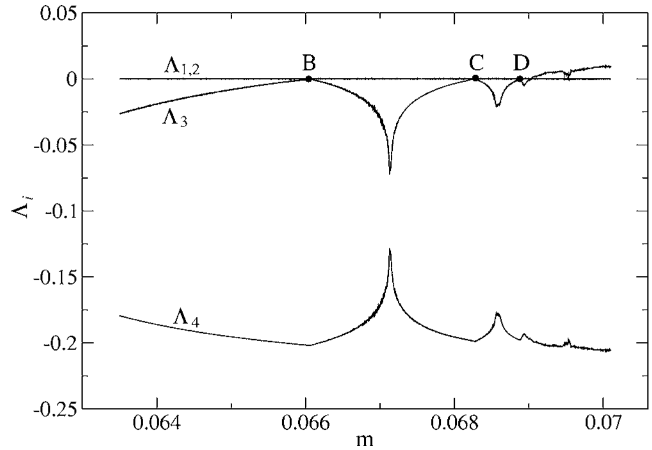

FIG. 6. Dependence of the Lyapunov exponents on the parameter m (d=0.001, γ=0.2, g=0.5) in the region of torus stability between the lines l_t and l_u. (B, C, and D are points of torus doubling bifurcations).

spectra of motions clearly indicate period-doubling bifurcations. In the considered case the modulation undergoes the period-doubling bifurcation (period-doubling bifurcation of a cycle on the Poincaré map).

From a viewpoint of bifurcation theory, two mechanisms of torus doubling bifurcation can be realized in this case.

• In the vicinity of bifurcation point the resonance phenomenon on the torus takes place and as a result a resonant limit cycle emerges. Then the resonant cycle undergoes a period-doubling bifurcation. Finally, a doubled torus is born on the base of a new cycle.

• The period-doubling bifurcation of ergodic torus is not followed by resonance phenomena.

It is quite important to find out which of the two possible mechanisms of doubled torus birth takes place in the case being considered. For this purpose we calculate the spectrum of Lyapunov exponents (Λ_1–Λ_4) when crossing the points of torus doubling bifurcations.

As it is shown on Fig. 6, in the bifurcation points B, C, and D simultaneously the three largest Lyapunov exponents are equal to zero ($\Lambda_1=\Lambda_2=\Lambda_3=0$). The bifurcational mechanism is determined by the following changes in the signature of the third Lyapunov exponent spectrum:

$$\underset{T_2}{0,0,-,-} \quad \Rightarrow \quad \underset{T_3}{0,0,0,-} \quad \Rightarrow \quad \underset{T_2}{0,0,-,-}.$$

In numerical experiments, the parameter m was varied with a very small step $\Delta m = 3\times 10^{-6}$. Our calculations indicate that while crossing the bifurcation point, a limit cycle birth (the Lyapunov exponent spectrum: $0,-,-,-$) is not observed. These results again confirm the results presented in [6]. The ergodic torus undergoes a period-doubling bifurcation; resonance cycles are not observed in the numerical experiment.

The numerical results considered above demonstrate convincingly that depending on values of the controlling parameters, system (5) in the autonomous regime can realize stable two-frequency oscillations with different values of the winding number, as well as demonstrate a cascade of period-doubling bifurcations of a two-dimensional torus, and a transition to chaos through torus destruction.

IV. SYNCHRONIZATION OF QUASIPERIODIC MOTIONS

By means of this generator of quasiperiodic oscillations, one can treat a new problem in the theory of synchronization: the synchronization of a two-dimensional torus. That problem is a natural generalization from the problem of synchronization of a limit cycle to the case of interaction of two generators with quasiperiodic motions. We start with the case of external synchronization of one such generator.

A. External synchronization

We consider the case of unidirectional forcing of the quasiperiodic motions of one generator on another, which also demonstrates a quasiperiodic behavior. The characteristic frequencies of both generators are not identical. The equations of that system of two interacting generators (5) with unidirectional coupling have the form

$$\dot{x}_1 = mx_1 + y_1 - x_1\varphi_1 - dx_1^3 + Kx_2,$$

$$\dot{y}_1 = -x_1,$$

$$\dot{z}_1 = \varphi_1,$$

$$\dot{\varphi}_1 = -\gamma\varphi_1 + \gamma\Phi(x_1) - gz_1,$$

$$\dot{x}_2 = mx_2 + y_2 - x_2\varphi_2 - dx_2^3,$$

$$\dot{y}_2 = -x_2,$$

$$\dot{z}_2 = \varphi_2,$$

$$\dot{\varphi}_2 = -\gamma\varphi_2 + \gamma\Phi(x_2) - g_1 z_2. \tag{6}$$

We consider the influence of the second generator on the first one (additive term Kx_2 in the first equation), the intensity of which is determined by the value of coupling strength K. For the fixed values of parameters m=0.06, d=0.001, γ=0.2, and g=0.55, the first generator in the autonomous regime demonstrates quasiperiodic motions with the winding number value $\theta_1 = f_{11}/f_{01}$. The winding number of the second generator θ_2 is controlled by the parameter g_1. In the case of $g \neq g_1$ both winding numbers are different, $\theta_1 \neq \theta_2$. Like in case of synchronization of a limit cycle for which the frequency mismatch $\Delta f = f_{01} - f_{02}$ is introduced, we consider here the winding number mismatch $\Delta\theta \simeq g - g_1$. We observe oscillatory regimes on the plane of the controlling parameters K and g_1. The numerical results are shown in Fig. 7, where the structure of synchronization regions is uncovered. In the large "tongue" of synchronization, which is bounded by the lines l_c, the basic frequencies of quasiperiodic motions appear to be locked, $f_{01} = f_{02}$. With this, the modulation frequencies remain different, i.e., $f_{11} \neq f_{12}$. Hence, the phenomenon of partial synchronization of quasiperiodic motions

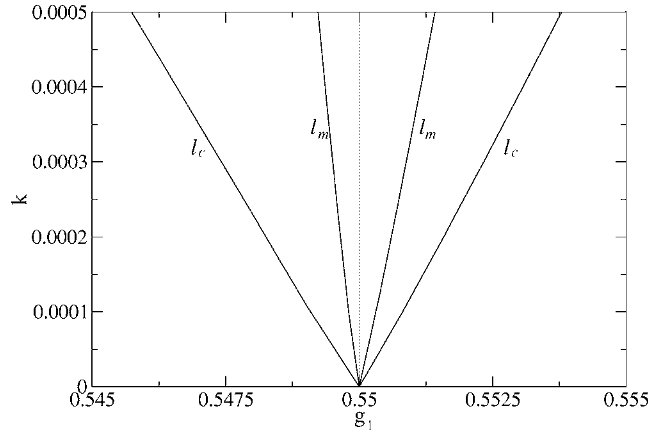

FIG. 7. Regions of synchronization of the carrying (l_c) and the modulation (l_m) frequencies of system (6) (m=0.06, γ=0.2, g=0.55, d=0.001).

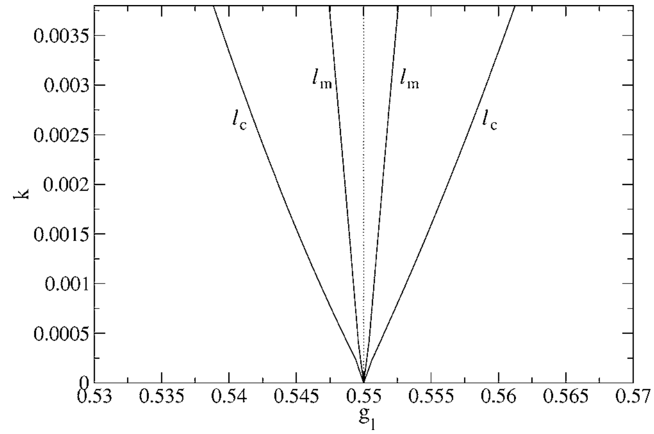

FIG. 9. Synchronization regions of the carrying (l_c) and the modulation (l_m) frequencies of system (7) (m=0.06, γ=0.2, g=0.55, d=0.001).

takes place. In the narrower region bounded by the bifurcational lines l_m, locking of the modulation frequencies $f_{11}=f_{12}$ and, correspondingly, locking of the winding numbers is observed, $\theta_1=\theta_2$. This phenomenon is also illustrated in Fig. 8. One can see that there exists a bounded region of values of the winding number mismatch Δg_1, in which $\theta_2/\theta_1=1$. Hence, the winding number of the second generator locks the winding number of the first generator. As for limit cycles, the width of a region of winding number locking increases as the forcing intensity increases, i.e., by the parameter K.

B. Mutual synchronization

As expected, the phenomenon of winding number locking is also observed in the case when two generators of quasiperiodic motions are mutually and symmetrically coupled. In that case the equations have the following form:

$$\dot{x}_1 = mx_1 + y_1 - x_1\varphi_1 - dx_1^3 + K(x_2 - x_1),$$

$$\dot{y}_1 = -x_1,$$

$$\dot{z}_1 = \varphi_1,$$

$$\dot{\varphi}_1 = -\gamma\varphi_1 + \gamma\Phi(x_1) - gz_1,$$

$$\dot{x}_2 = mx_2 + y_2 - x_2\varphi_2 - dx_2^3 + K(x_1 - x_2),$$

$$\dot{y}_2 = -x_2,$$

$$\dot{z}_2 = \varphi_2,$$

$$\dot{\varphi}_2 = -\gamma\varphi_2 + \gamma\Phi(x_2) - g_1 z_2. \tag{7}$$

In the numerical experiments the following parameters are fixed: m=0.06; γ=0.2; g=0.55; d=0.001. We vary the coupling strength K and the parameter g_1 that determines the winding numbers θ_1 and θ_2 mismatch. The results of our computations are shown in Figs. 9 and 10. The structure of synchronization regions (Fig. 9) is similar to those shown for the case of external synchronization (Fig. 7), but it is more symmetric with respect to the mismatch. Like in the case of external synchronization (Fig. 8), the phenomenon of wind-

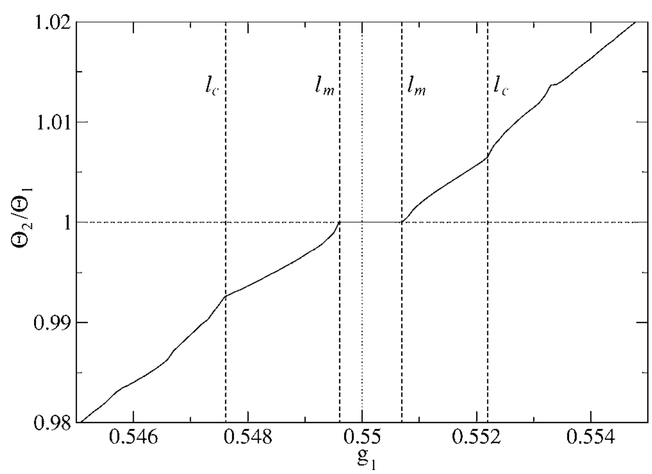

FIG. 8. Phenomenon of winding number locking (m=0.06, γ=0.2, g=0.55, d=0.001, k=0.0003).

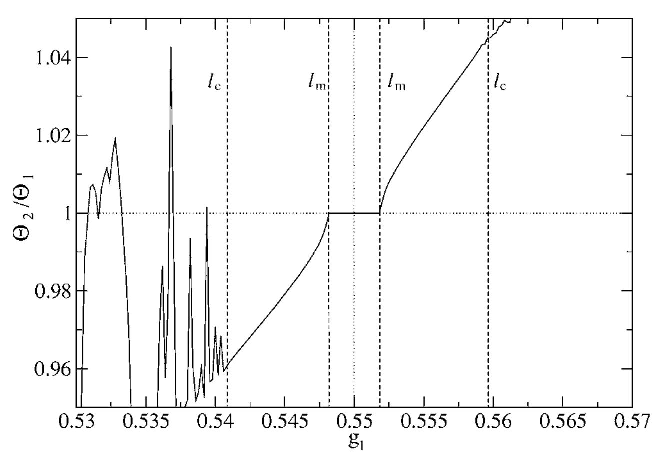

FIG. 10. Effect of winding number locking (m=0.06, γ=0.2, g=0.55, d=0.001, k=0.003).

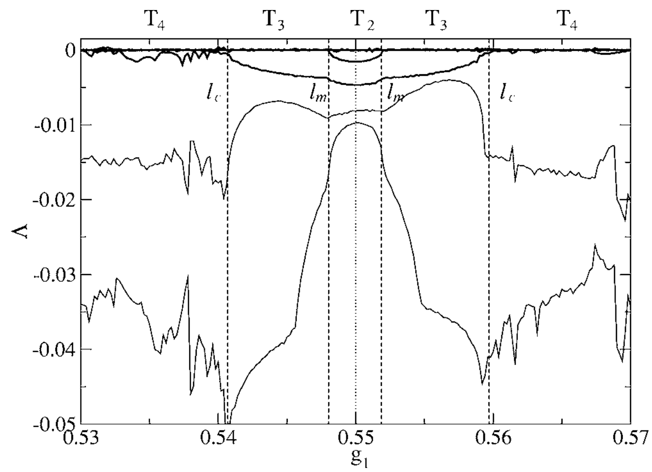

FIG. 11. Evolution of the Lyapunov exponent spectrum when the parameter g_1 is varied ($m=0.06$, $\gamma=0.2$, $g=0.55$, $d=0.001$, $k=0.003$). $\Lambda_{7,8} \ll 0$ are not shown.

ing number locking is also observed (Fig. 10).

In both cases of synchronization (external and mutual) we have studied that phenomenon from the point of view of frequency locking. The effect of phase synchronization was not considered. All the observed phenomena (Figs. 7–10) are stable and have sufficiently large basins of attraction. The results shown were computed for the following initial conditions: $x_1=-1.367\,709$, $y_1=1.313\,016$, $z_1=0.753\,935$, $\varphi_1=-0.771\,030$, $x_2=-1.367\,409$, $y_2=1.312\,016$, $z_2=0.754\,935$, $\varphi_2=-0.772\,030$).

The results shown in Figs. 7–10 were obtained by analyzing in detail time series and power spectra of oscillations. The phenomenon of partial and complete locking can be identified as a coincidence of basic and modulation frequencies in the power spectra of oscillations, but as there is a large number of additional frequencies besides the basic ones, that method does not give exact results. The frequencies of motions were determined by analyzing the time series. In this connection it is interesting to include another characterization—the evolution of the full spectrum of Lyapunov exponents as one of the controlling parameters is varied. Figure 11 represents the dependence of the Lyapunov exponents on the parameter g_1, correspondingly to Fig. 10.

Outside the region of synchronization, the Lyapunov exponent spectrum includes four largest exponents equal to zero

$$0,0,0,0,-,-,-,- .$$

When crossing the line l_c, there remain only three zero Lyapunov exponents

$$0,0,0,-,-,-,-,- .$$

In the region of winding number locking, there are only two Lyapunov exponents being equal to zero

$$0,0,-,-,-,-,-,- .$$

Outside the region of synchronization, system (7) demonstrates the regime of a four-dimensional torus T_4. That regime can be called the regime of beating, which takes place in the case when two two-dimensional tori with different winding numbers interact. On crossing the line l_c inside the region of synchronization, the transition $T_4 \rightarrow T_3$ takes place, and a three-dimensional torus emerges. Next, after crossing the line l_m a two-dimensional torus appears $T_3 \rightarrow T_2$. One can suppose (by analogy with synchronization of limit cycles), that the transition $T_4 \rightarrow T_3$ corresponds to the resonance phenomenon of the three-dimensional torus on the four-dimensional one. At that, two of the four frequencies f_{01}, f_{11} and f_{02}, f_{12} are locked ($f_{01}=f_{02}$), and three frequencies remain independent. The transition $T_3 \rightarrow T_2$ corresponds to a stronger resonance when a winding number locking takes place. Perhaps, this phenomenon corresponds to the existence of a resonant two-dimensional torus which lies on the surface of the four-dimensional one. Note that we have so far no proof for the suppositions made. Nevertheless, the detailed analysis of the full spectrum of Lyapunov exponents can be used to diagnose the regimes of synchronization of quasiperiodic motions.

V. CONCLUSIONS

A generator of two-dimensional oscillations has been introduced, for which a mathematical model has been developed, too. The model serves as an autonomous dynamical system in R^4 and has four controlling parameters (5). Depending on values of the parameters the generator demonstrates the regimes of periodic motions, period-doubling bifurcations of limit cycles, soft bifurcations leading to the birth of a two-dimensional torus, torus doubling bifurcations, and transitions to chaos through the two-dimensional torus stability loss. By means of the model of generator (5) the phenomena of external and mutual synchronization of quasiperiodic motions with two independent frequencies have been investigated. With this, a new phenomenon—winding number locking on a two-dimensional torus—has been revealed. Our additional numerical experiments have also shown that the observed phenomenon is stable under the influence of external additive noise of intensity $D=5\times10^{-9}$.

The model under consideration (5) may serve as one of the simplest paradigmatic models of nonlinear dynamics for studying properties of two-frequency quasiperiodic oscillations.

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[1] D. Ruelle and F. Takens, Commun. Math. Phys. **20**, 167 (1971).
[2] S. Newhouse, D. Ruelle, and F. Takens, Commun. Math. Phys. **64**, 35 (1979).
[3] R. Tavakol and A. Tworkowski, Phys. Lett. **100**(6), 273 (1984); , **111**(7), 317 (1985).
[4] L. Landau, Dan. Tidsskr. Farm. **44**, 339 (1944) (in Russian).
[5] E. Hopf, Commun. Pure Appl. Math. **1**, 303 (1948).
[6] V. Anishchenko, T. Letchford, and M. Safonova, Izv. Vyssh. Uchebn. Zaved., Radiofiz. **28**(9), 1112 (1985) (in Russian) (also presented in [11]).
[7] K. Kaneko, *Collapse of Tori and Genesis of Chaos in Dissipative Systems* (World Scientific, Singapore, 1986).
[8] V. Franceschini, Physica D **3**, 285 (1983).
[9] U. Feudel, M. Safonova, J. Kurths, and V. Anishchenko, Int. J. Bifurcation Chaos Appl. Sci. Eng. **6**, 1319 (1996).
[10] V. Anishchenko, M. Safonova, U. Feudel, and J. Kurths, Int. J. Bifurcation Chaos Appl. Sci. Eng. **4**, 595 (1994).
[11] V. Anishchenko, *Dynamical Chaos – Models and Experiments.* (World Scientific, Singapore, 1995).
[12] T. Matsumoto, L. Chua, and R. Tokunada, IEEE Trans. Circuits Syst. CAS-34, **3**, (1987).
[13] L. Shilnikov, Int. J. Bifurcation Chaos Appl. Sci. Eng. **7**(9), 1953 (1997).
[14] V. Anishchenko and S. Nikolaev, Tech. Phys. Lett. **31**, 88 (2005).